(Fibonacci seq.)

$$T(n) = T(n-1) + T(n-2) \geq \left(\frac{3}{2}\right)^n$$

book says this holds for $n > 4$

assume so, show it holds for $n+1$ (inductive proof)

$$T(n+1) = T(n) + T(n-1)$$

$$T(n) \geq \left(\frac{3}{2}\right)^n$$

$$T(n+1) = \left(\frac{3}{2}\right)^n + \left(\frac{3}{2}\right)^{n-1}$$

show $\geq \left(\frac{3}{2}\right)^{n+1}$

$$\left(\frac{3}{2}\right)^n + \left(\frac{3}{2}\right)^{n-1}$$

want this to be

$$\geq \left(\frac{3}{2}\right)^{n+1}$$

$$\left(\frac{3}{2}\right)^{-1}\left(\frac{3}{2}\right)^{n+1}$$

$$\left(\frac{3}{2}\right)^{-1}\left(\frac{3}{2}\right)^{-1}\left(\frac{3}{2}\right)^{n+1}$$

$$\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)^{n+1} + \left(\frac{2}{3}\right)^2\left(\frac{3}{2}\right)^{n+1}$$

$$= \left(\frac{2}{3}+\frac{4}{9}\right)\left(\frac{3}{2}\right)^{n+1}$$

$$= \left(\frac{10}{9}\right)\left(\frac{3}{2}\right)^{n+1} \geq \left(\frac{3}{2}\right)^{n+1}$$

▣ Q.E.D.

- inductive proof: rule of thumb:

a) assume assertion holds for n

b) inductive step: show assertion holds for $n+1$

- generally, we deal with recursive calls that split lists up in two, then recursion operates on each half.

$T(1) = 1$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$T(n)$: which $O(f(n))$ class this falls in

$2T\left(\frac{n}{2}\right)$: 2 x time to process sublists

n: time to split into sublists

solution: 2 methods

telescoping sum

recurrence back substitution

e.g., telescoping sum

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

divide by n — reduce to constant

$$\frac{T(n)}{n} = \frac{2T\left(\frac{n}{2}\right)}{n} + 1$$

$$= \frac{T\left(\frac{n}{2}\right)}{\frac{n}{2}} + 1$$

$$\frac{T(n)}{n} = \frac{T(n/2)}{n/2} + 1$$ } valid for any n a power of 2

$$\frac{T(n/2)}{n/2} = \frac{T\left(\frac{n}{4}\right)}{n/4} + 1$$

$$\frac{T(n/4)}{n/4} = \frac{T\left(\frac{n}{8}\right)}{n/8} + 1$$

$$\vdots$$

$$\frac{T(2)}{2} = \frac{T(1)}{1} + 1$$

---

add it all up (tele. sums)

Sum:

LHS:

$$\frac{T(n)}{n} + \frac{T(n/2)}{n/2} + \cdots + \frac{T(2)}{2}$$

$$=$$

RHS:

$$\frac{T(\frac{n}{2})}{n/2} + 1 + \frac{T(n/4)}{n/4} + 1 + \cdots + \frac{T(1)}{1} + 1$$

$$\frac{T(n)}{n} = \sum_{\log_2 n} 1 + \frac{T(1)}{1}$$

$$\frac{T(n)}{n} = \sum_{\lg n} 1 + \frac{T(1)}{1}$$

mult by n to get $T(n)$

$$T(n) = n \sum_{\lg n} 1 + n(1)$$

$$= n \lg n + n$$

$$\in O(n \lg n)$$

Method 2: recurrive back sub.

$T(1) = 0$

$T(n) = 2T\left(\frac{n}{2}\right) + n - 1$

---

level 1: $T(n) = 2T\left(\frac{n}{2}\right) + n - 1$

$= n - 1 + 2T\left(\frac{n}{2}\right)$

---

level 2: with $n = n/2$

$= n - 1 + 2\left(\frac{n}{2} - 1 + 2T\left(\frac{n}{4}\right)\right)$

$= n - 1 + n - 2 + 4T\left(\frac{n}{4}\right)$

Level 3: $n-1+2T\left(\frac{n}{2}\right)$ $n/4$

$= n-1+n-2+4\left(\frac{n}{4}-1+2T\left(\frac{n}{8}\right)\right)$

$= n-1+n-2+n-4+8T\left(\frac{n}{8}\right)$

Level 4:

$= n-1+n-2+n-4+n-8+16T\left(\frac{n}{16}\right)$

Level K:

$$= \sum_{i=0}^{K-1} n - \sum_{i=0}^{K-1} 2^i + 2^K T\left(\frac{n}{2^K}\right)$$

$$T(n) = \sum_{i=0}^{k-1} n - \sum_{i=0}^{k-1} 2^i + 2^k T\left(\frac{n}{2^k}\right)$$

want $T(1)$

to get $T\left(\frac{n}{2^k}\right)$ to be $T(1)$

let $2^k = n$ so that $T\left(\frac{n}{n}\right) = T(1)$

if $2^k = n$, $k = \log_2 n$

$$T(n) = \sum_{i=0}^{k-1} n + \sum_{i=0}^{k-1} 1^i + nT(1)$$

$\sum_{i=0}^{k-1} n$ look this up

$\sum_{i=0}^{n} k = k(n+1)$

$\rightarrow n(k-1+1) = nk$

$\sum_{i=0}^{k-1} 2^i$ look this up

$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

$\rightarrow 2^{(k-1+1)} - 1 = 2^k - 1$

$$T(n) =$$

$$nk - (2^k - 1) + 2^k \cancel{T(1)}$$

$$\| \; 0$$

but $2^k = n$

$$k = \log_2 n$$

$$T(n) = n \log_2 n - n + 1$$

$$\in O(n \lg n)$$

Handy formulae:

$$\sum_{i=0}^{n} k = k(n+1)$$

$$1+2+3+\cdots+n$$

$$n+(n-1)+\cdots+2+1$$

$$\frac{(n+1)n}{2}$$

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

How did $T(n)$ come about?

Counting up # of statements in alg. pseudocode, i.e.,

```
insert(list)
{
    if(list.size()==1) } O(1)
        return;
    //split list in 2   } O(n)
    insert(list/2);     } O(n/2)
    insert(list/2);     } O(n/2)
}
```

$O(1)$

$O(n)$

$O(\frac{n}{2})$

$O(\frac{n}{2})$

$$T(n) = 2T(\frac{n}{2}) + n + 1$$

$$T(1) = 1$$