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# Speeded Up Detection of Squared Fiducial Markers

Francisco J. Romero-Ramirez<sup>1</sup>, Rafael Muñoz-Salinas<sup>1,2,\*</sup>, Rafael Medina-Carnicer<sup>1,2</sup>

#### Abstract

Squared planar markers have become a popular method for pose estimation in applications such as autonomous robots, unmanned vehicles or virtual trainers. The markers allow estimating the position of a monocular camera with minimal cost, high robustness, and speed. One only needs to create markers with a regular printer, place them in the desired environment so as to cover the working area, and then registering their location from a set of images.

Nevertheless, marker detection is a time-consuming process, especially as the image dimensions grows. Modern cameras are able to acquire high resolutions images, but fiducial marker systems are not adapted in terms of computing speed. This paper proposes a multi-scale strategy for speeding up marker detection in video sequences by wisely selecting the most appropriate scale for detection, identification and corner estimation. The experiments conducted show that the proposed approach outperforms the state-of-the-art methods without sacrificing accuracy or robustness. Our method is up to 40 times faster than the state-of-the-art method, achieving over 1000 fps in 4K images without any parallelization.

Keywords: Fiducial Markers, Marker Mapping, SLAM.

# 1. Introduction

Pose estimation is a common task for many applications 2 such as autonomous robots [1, 2, 3], unmanned vehicles 3 [4, 5, 6, 7, 8] and virtual assistants [9, 10, 11, 12], among 4 other.

Cameras are cheap sensors that can be effectively used for this task. In the ideal case, natural features such as 7 keypoints or texture [13, 14, 15, 16] are be employed to 8 create a map of the environment. Although some of the 9 traditional problems of previous methods for this task have 10 been solved in the last few years, other problems remain. 11 For instance, they are subject to filter stability issues or 12 significant computational requirements. 13

In any case, artificial landmarks are a popular approach 14 for camera pose estimation. Square fiducial markers, com-15 prised by an external squared black border and an internal 16 identification code, are especially attractive because the 17 camera pose can be estimated from the four corners of a 18 single marker [17, 18, 19, 20]. The recent work of [21] is 19

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a step forward the use of this type of markers in largescale problems. One only need to print the set of markers with a regular printer, place them in the area under which the camera must move, and take a set of pictures of the markers. The pictures are then analyzed and the threedimensional marker locations automatically obtained. Afterward, a single image spotting a marker is enough to estimate the camera pose.

Despite the recent advances, marker detection can be a time-consuming process. Considering that the systems requiring localization have in many cases limited resources, such as mobile phones or aerial vehicles, the computational effort of localization should be kept to a minimum. The computing time employed in marker detection is a function of the image size employed: the larger the images, the slower the process. On the other hand, high-resolution images are preferable since markers can be detected, even if far from the camera, with high accuracy. The continuous reduction in the cost of the cameras, along with the increase of their resolution, makes necessary to develop methods able to reliably detect the markers in highresolution images.

The main contribution of this paper is a novel method 42 for detecting square fiducial markers in video sequences. The proposed method relies on the idea that markers can 44 be detected in smaller versions of the image, and employs a 45 multi-scale approach to speed up computation while maintaining the precision and accuracy. In addition, the sys-47 tem is able to dynamically adapt its parameters in order 48

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to achieve maximum performance in the analyzed video sequence. Our approach has been extensively tested and compared with the state-of-the-art methods for marker detection. The results show that our method is more than an order of magnitude faster than state-of-the-art approaches without compromising robustness or accuracy, and without requiring any type of parallelism.

The remainder of this paper is structured as follows. Section 2 explains the works most related to ours. Section 3 details our proposal for speeding up the detection of markers. Finally, Section 4 gives a exhaustive analysis of the proposed method and Section 5 draws some conclusions.

## 62 2. Related works

Fiducials marker systems are commonly used for camera 63 localization and tracking when robustness, precision, and 64 speed are required. In the simplest case, points are used 65 as fiducial markers, such as LEDs, retroreflective spheres 66 or planar dots [22, 23]. However, their main drawback is 67 the need of a method to solve the assignment problem, i.e., 68 assigning a unique and consistent identifier to each element 69 over time. In order to ease the problem, a common solution 70 consists in adding an identifying code into each marker. 71 Examples of this are planar circular markers [24, 25], 2D-72 barcodes [26, 27] and even some authors have proposed 73 markers designed using evolutionary algorithms [28]. 74

Amongst all proposed approaches, these based on squared planar markers have gained popularity. These markers consist of an external black border and an internal code (most often binary) that uniquely identifies each marker (see Fig 1). Their main advantage is that the pose of the camera can be estimated from a single marker.

ARToolKit [29] is one of the pioneer proposals. They 81 employed markers with a custom pattern that is identified 82 by template matching. This identification method, how-83 ever, is prone to error and not very robust to illumination 84 changes. In addition, the method's sensitivity degrades 85 as the number of markers increases. As a consequence, 86 other authors improved that work by using binary BCH 87 codes [30] (which allows a more robust error detection) and 88 named it ARToolKit+ [31]. The project was halted and 89 followed by the Studierstube Tracker project [32], which is 90 privative. Similar to the ARToolKit+ project is the dis-91 continued project ARTag [33]. 92

BinARyID [34] is one of the first systems that proposed a method for generating customizable marker codes. Instead of using a predefined set of codes, they proposed a method for generating the desired number of codes for each particular application. However, they do not consider the possibility of error detection and correction. AprilTags [18], however, proposed methods for error detection and correction, but their approach was not suitable for a large number of markers.

The work ArUco [17] is probably the most popular sys-102 tem for marker detection nowadays. It adapts to non-103 uniform illumination, and is very robust, being able to 104 do error detection and correction of the binary codes im-105 plemented. In addition, the authors proposed a method 106 to obtain optimal binary codes (in terms of intermarker-107 distance) using Mixed Integer Linear Programming [35]. 108 Chilitags [36] is a variation of ArUco that employs a sim-109 pler method for decoding the marker binary codes. As we 110 show in the experimental section, the method has a bad 111 behavior in high-resolution images. 112

The recent work [21] is a step towards the applicabil-113 ity of such methods to large areas, proposing a method 114 for estimating the three-dimensional location of a set of 115 markers freely placed in the environment (Fig 1). Given 116 a set of images taken with a regular camera (such as a 117 mobile phone), the method automatically estimates their 118 location. This is an important step that allows extending 119 the robust localization of fiducial markers to very large 120 areas. 121

Although all fiducial marker systems aim maximum 122 speed in their design, few specific solutions have been pro-123 posed to speed up the detection process. The work of 124 Johnston et. al. [37] is an interesting example in which 125 the authors propose a method to speed up computation by 126 parallelizing the image segmentation process. Neverthe-127 less, both speed and computing power is a crucial aspect, 128 especially if the localization system needs to be embedded 129 in devices with limited resources. 130

Our work can be seen as an improvement of the ArUco 131 system, that according to our experience, is one of the most 132 reliable fiducial marker systems nowadays (see Sec 4 for 133 further details). We propose a novel method for marker de-134 tection and identification that allows to speed up the com-135 puting time in video sequences by wisely exploiting tempo-136 ral information and an applying multi-scale approach. In 137 contrast to previous works, no parallelization is required in 138 our method, thus making it especially attractive for mobile 139 devices with limited computational resources. 140

# 3. Speeded up marker detection

This section provides a detailed explanation of the method proposed for speeding up the detection of squared planar markers. First, Sect. 3.1 provides an overview of the pipeline employed in the previous work, ArUco [17], for marker detection and identification, highlighting the parts of the process susceptible to be accelerated. Then, 147



Figure 1: Detection and identification pipeline of ArUco. (a) Original image. (b) Image thresholded using an adaptive method. (c) Contours extracted. (d) Filtered contours that approximate to four-corner polygons. (e) Canonical image computed for one of the squared contours detected. (f) Binarization after applying Otsu's method.

Sect. 3.2 explains the proposed method to speed up the
process.

# <sup>150</sup> 3.1. Marker detection and identification in ArUco

The main steps for marker detection and identification proposed in ArUco [17] are depicted in Figure 1. Given the input image I (Figure 1a), the following steps are taken:

• Image segmentation (Figure 1b). Since the designed 154 markers have an external black border surrounded by 155 a white space, the borders can be found by segmen-156 tation. In their approach, a local adaptive method is 157 employed: the mean intensity value m of each pixel 158 is computed using a window size  $w_t$ . The pixel is set 159 to zero if its intensity is greater than m-c, where c 160 is a constant value. This method is robust and ob-161 tains good results for a wide range of values of its 162 parameters  $w_t$  and c. 163

- Contour extraction and filtering (Figures 1(c,d)). The 164 contour following algorithm of Suzuki and Abe [38]165 is employed to obtain the set of contours from the 166 thresholded image. Since most of the contours ex-167 tracted correspond to irrelevant background elements, 168 a filtering step is required. First, contours too small 169 are discarded. Second, the remaining contours are 170 approximated to its most similar polygon using the 171 Douglas and Peucker algorithm [39]. Those that do 172 not approximate well to a four-corner convex polygon 173 are discarded from further processing. 174
- Marker code extraction (Figures 1(e,f)). The next step consists in analyzing the inner region of the re-

maining contours to determine which of them are valid 177 markers. To do so, perspective projection is first re-178 moved by computing the homography matrix, and the 179 resulting canonical image (Fig. 1e) is thresholded us-180 ing the Otsu's method [40]. The binarized image 181 (Fig. 1f) is divided into a regular grid and each ele-182 ment is assigned a binary value according to the ma-183 jority of the pixels in the cell. For each marker candi-184 date, it is necessary to determine whether it belongs 185 to the set of valid markers or if it is a background el-186 ement. Four possible identifiers are obtained for each 187 candidate, corresponding to the four possible rota-188 tions of the canonical image. If any of the identifiers 189 belong to the set of valid markers, then it is accepted. 190

• Subpixel corner refinement. The last step consists in 191 estimating the location of the corners with subpixel 192 accuracy. To do so, the method employs a linear 193 regression of the marker's contour pixels. In other 194 words, it estimates the lines of the marker sides em-195 ploying all the contour pixels and computes the in-196 tersections. This method, however, is not reliable for 197 uncalibrated cameras with small focal lenses (such as 198 fisheye cameras) since they usually exhibit high dis-199 tortion. 200

When analyzing the computing times of this pipeline, 201 it can be observed that the Image segmentation and the 202 Marker code extraction steps are consuming most of the 203 computing time. The time employed in the image segmen-204 tation step is proportional to the image size, that also in-205 fluences the length of the contours extracted and thus the 206 computing time employed in the Contour extraction and 207 filtering step. The extraction of the canonical image (in 208 the Marker code extraction step) involves two operations. 209 First, computing the homography matrix, which is cheap. 210 But then, the inner region of each contour must be warped 211 to create the canonical image. This step requires access to 212 the image pixels of the contour region performing an inter-213 polation in order to obtain the canonical image. The main 214 problem is that the time required to obtain the canonical 215 image depends on the size of the observed contour. The 216 larger a contour in the original image, the more time it is 217 required to obtain the canonical image. Moreover, since 218 most of the contours obtained do not belong to markers. 219 the system may employ a large amount of time computing 220 canonical images that will be later rejected. 221

A simpler approach to solving that problem would be to directly sample a few sets of pixels from the inner region of the marker. This is the method employed in ChiliTags. However, as it will be shown in the experimental section, it is prone to many false negatives. 226



Figure 2: **Process pipeline** Main steps for fast detection and identification of squared planar markers.(a) Original input image. (b) Resized image for marker search. (c) Thresholded image. (d) Rectangles found (pink). (e) Markers detected with its corresponding identification. The image pyramid is used to speed up homography computation. (f) The corners obtained in (e) are upsampled to find their location in the original image with subpixel precision.

#### 227 3.2. Proposed method

The key ideas of our proposal in order to speed up the 228 computation are explained below. First, while the adap-229 tive thresholding method employed in ArUco is robust to 230 many illumination conditions without altering its param-231 eters, it is a time-consuming process that requires a con-232 volution. By taking advantage of temporal information, 233 the adaptive thresholding method is replaced by a global 234 thresholding approach. 235

Second, instead of using the original input image, a 236 smaller version is employed. This is based on the fact 237 that, in most cases, the useful markers for camera pose 238 estimation must have a minimum size. Imagine an image 239 of dimensions  $1920 \times 1080$  pixels, in which a marker is de-240 tected as a small square with a side length of 10 pixels. 241 Indeed, the estimation of the camera pose is not reliable 242 at such small resolution. Thus, one might want to set a 243 minimum length to the markers employed for camera pose 244 estimation. For instance, let say that we only use markers 245 with a minimum side length of  $\dot{\tau}_i = 100$  pixels, i.e., with a 246 total area of 10.000 pixels. Another situation in which we 247 can set a limit to the length of markers is when processing 248 video sequences. It is clear that the length of a marker 249 must be similar to its length in the previous frame. 250

Now, let us also think about the size of the canonical 251 images employed (Figure 1e). The smaller the image, the 252 faster the detection process but the poorer the image qual-253 ity. Our experience, however, indicates that very reliable 254 detection of the binary code can be obtained from very 255 small canonical images, such  $32 \times 32$  pixels. In other words, 256 all the rectangles detected in the image, no matter their 257 side length, are reduced to canonical images of side length 258  $\tau_c = 32$  pixels, for the purpose of identification. 259

Our idea, then, is to employ a reduced version of the <sup>260</sup> input image, using the scale factor  $\frac{\tau_c}{\tau_i}$ , so as to speed up <sup>261</sup> the segmentation step. In the reduced image, the smallest <sup>262</sup> allowed markers, with a side length of 100 pixels in the <sup>263</sup> original image, appears as rectangles with a side length of <sup>264</sup> 32 pixels. As a consequence, there will be no loss of quality <sup>265</sup> when they are converted into the canonical image. <sup>266</sup>

This idea has one drawback: the location of the corners 267 extracted in the low resolution image is not as good esti-268 mations as the ones that can be obtained in the original 269 image. Thus, the pose estimated with them will have a 270 higher error. To solve that problem, a corner upsampling 271 step is included, in which the precision of the corners is re-272 fined up to subpixel accuracy in the original input image 273 by employing an image pyramid. 274

Finally, it must be considered that the generation of the canonical image is a very time-consuming operation (even if the process is done in the reduced image) that proportional to the contour length. We propose a method to perform the extraction of the canonical images in almost constant time (independently of the contour length) by wisely employing the image pyramid. 281

Below, there is a detailed explanation of the main steps of the proposed method, using Figure 2 to ease the explanation.

1. Image Resize: Given the input image I (Fig 2a), the first step consists in obtaining a resized version  $I^r$  (Fig 2b) that will be employed for segmentation. As previously pointed out, the size of the reduced image is calculated as:

$$I_w^r = \left\lfloor \frac{\tau_c}{\dot{\tau}_i} I_w \right\rfloor; I_h^r = \left\lfloor \frac{\tau_c}{\dot{\tau}_i} I_h \right\rfloor, \tag{1}$$

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where the subscripts w and h denotes width and height



Figure 3: Pyramidal Warping. Scene showing tree marker at different resolutions. The left column shows the canonical images warped from the pyramid of images. Larger markers are warped from smaller images. For each marker, the image of the pyramid that minimizes the warping time while preserving the resolution is selected.

respectively. In order to decouple the desired minimum marker size from the input image dimensions, we define  $\dot{\tau}_i$  as:

$$\dot{\tau}_i = \tau_c + max(I_w, I_h)\tau_i \mid \tau_i \in [0, 1],$$
 (2)

where the normalized parameter  $\tau_i$  indicates the minimum marker size as a value in the range [0, 1]. When  $\tau_i = 0$ , the reduced image will be the same size as the original image. As  $\tau_i$  tends to one, the image  $I^r$  becomes smaller, and consequently, the computational time required for the following step is reduced. The impact of this parameter in the final speed up is measured in the experimental section.

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2. Image Segmentation: As already indicated, a global 293 threshold method is employed using the following pol-294 icy. If no markers were detected in the previous frame, 295 a random threshold search is performed. The random 296 process is repeated up to three times using the range 297 of threshold values [10, 240]. For each tested thresh-298 old value, the whole pipeline explained below is per-299 formed. If after a number of attempts, no marker is 300 found, it is assumed that no markers are visible in the 301 frame. If at least one marker is detected, a histogram 302 is created using the pixel values of all detected mark-303 ers. Then, Otsu's algorithm [40] is employed to select 304 the optimal threshold for the next frame. The calcu-305

lated threshold is applied to  $I^r$  in order to obtain  $I^t$ 306 (Fig 2c). As we show experimentally, the proposed 307 method can adapt to smooth and abrupt illumination 308 changes. 309

- 3. Contour Extraction and Filtering: First, contours are 310 extracted from the image  $I^t$  using Suzuki and Abe al-311 gorithm [38], then small contours are removed. Since 312 the extracted contours will rarely be squared (due to 313 perspective projection), their perimeter is employed 314 for rejection purposes: those with a perimeter smaller 315 than  $P(\tau_c) = 4 \times \tau_c$  pixels are rejected. For the re-316 maining contours, a polygonal approximation is per-317 formed using Douglas and Peucker algorithm [39], and 318 those that do not approximate to a convex polygon of 319 four corners are also rejected. Finally, the remaining 320 contours are the candidates to be markers (Fig 2d). 321
- 4. Image Pyramid Creation: An image pyramid

$$\mathcal{I} = (I^0, \dots, I^n)$$

with a set of resized versions of I, is created.  $I^0$  de-322 notes the original image and the subsequent images 323  $I^i$  are created by subsampling  $I^{i-1}$  by a factor of two. 324 The number n of images in the pyramid is such that the smallest image dimensions is close to  $\tau_c \times \tau_c$ , i.e.,

$$n = \underset{v \mid I^v \in \mathcal{I}}{\operatorname{argmin}} |(I^v_w I^v_h) - \tau^2_c|.$$
(3)

5. Marker Code Extraction: In this step the canonical 325 images of the remaining contours must be extracted 326 and then binarized. Our method uses the pyramid 327 of images  $\mathcal{I}$  previously computed to ensure that the 328 process is performed in constant time, independently 329 of the input image and contour sizes. The key princi-330 ple is selecting, for each contour, the image from the 331 pyramid in which the contour length is most similar 332 to the canonical image length  $P(\tau_c)$ . In this manner, 333 warping is faster. 334

Let us consider a detected contour  $\vartheta \in I^r$ , and denote by  $P(\vartheta)^j$  its perimeter in the image  $I^j \in \mathcal{I}$ . Then, the best image  $I^h \in \mathcal{I}$  for homography computation is selected as:

$$I^{h} \mid h = \operatorname*{argmin}_{j \in \{0,1,\dots n\}} |P(\vartheta)^{j} - P(\tau_{c})|.$$

$$(4)$$

The pyramidal warping method employed can be bet-336 ter understood in Fig. 3, which shows a scene with three markers at different distances. The left images represent the canonical images obtained while the right images show the pyramid of images. In our method, the canonical image of the smallest marker is extracted from the largest image in the pyramid (top

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Figure 4: **Test sequences.** (a) The set of 16 markers employed for evaluation. There are four markers from each method tested: ArUco, AprilTags, ArToolKit+ and ChiliTags. (b-e) Images from the video sequences used for testing. The markers are seen as small as in (b), and as big as in (e), where the marker represents the 40% of the total image area.

row of Fig 3). As the length of the marker increases,
smaller images of the pyramid are employed to obtain
the canonical view. This guarantees that the canonical image is obtained in almost constant time using
the minimum possible computation.

Finally, for each canonical image, the Otsu's method
[40] for binarization is employed, and the inner code
analyzed to determine whether it is a valid marker or
not. This is a very cheap operation.

6. Corner Upsampling: So far, markers have been de-352 tected in the image  $I^r$ . However, it is required to 353 precisely localize their corners in the original image 354 I. As previously indicated, the precision of the esti-355 mated camera pose is directly influenced by the pre-356 cision in the corner localization. Since the difference 357 in size between the images I and  $I^r$  can be very large, 358 a direct upsampling can lead to errors. Instead, we 359 proceed in incremental steps looking for the corners 360 in larger versions of the image  $I^r$  until the image I is 361 reached. 362

For the corner upsampling task, the image  $I^i \in \mathcal{I}$  of the pyramid with the most similar size to  $I^r$  is selected in the first place, i.e.,

$$I^{i} = \underset{I^{v} \in \mathcal{I}}{\operatorname{argmin}} |(I^{v}_{w}I^{v}_{h}) - (I^{r}_{w}I^{r}_{h})|.$$

$$(5)$$

Then, the position of each contour corner in the image  $I^i$  is computed by simply upsampling the corner locations. This is, however, an approximate estimation that does not precisely indicate the corner position in the image  $I^i$ . Thus, a corner refinement process is done in the vicinity of each corner so as to find its best location in the selected image  $I^i$ . For that purpose, the method implemented in the OpenCV library [41] 373 has been employed. Once the search is done in  $I^i$  for all corners, the operation is repeated for the image  $I^{i-1}$ , until  $I^0$  is reached. In contrast to the ArUco approach, this one is not affected by lens distortions. 377

7. Estimation of  $\tau_i$ : The parameter  $\tau_i$  has a direct influ-378 ence in the computation time. The higher it is, the 379 faster the computation. A naive approach consists 380 in setting a fixed value for this parameter. However, 381 when processing video sequences, the parameter can 382 be automatically adjusted at the end of each frame. 383 In the first image of the sequence, the parameter  $\tau_i$  is 384 set to zero. Thus, markers of any size are detected. 385 Then, for the next frame,  $\tau_i$  is set to a value slightly 386 smaller than the size of the smallest marker detected 387 in the previous frame. In this way, markers could be 388 detected even if the camera moves away from them. 380 Therefore, the parameter  $\tau_i$  can be dynamically up-390 dated as: 391

$$\tau_i = (1 - \tau_s) P(\vartheta^s) / 4 \tag{6}$$

where  $\vartheta^s$  is the marker with the smallest perimeter found in the image, and  $\tau_s$  is a factor in the range 393 (0,1] that accounts for the camera motion speed. For 394 instance, when  $\tau_s = 0.1$ , it means that in the next 395 frame,  $\tau_i$  is such that markers 10% smaller than the 396 smallest marker in the current image will be sought. 397 If no markers are detected in a frame,  $\tau_i$  is set to zero 398 so that in the next frame markers of any size can be 399 detected. 400 -2160p resolution---1080p resolution---720p resolution---600p resolution---480p resolution



Figure 5: **SpeedUp** of ArUco3 compared to ArUco, ArToolKit+, ChiliTags and AprilTags for resolutions:  $4K (3840 \times 2160)$ ,  $1080p (1920 \times 1080)$ ,  $720p (1280 \times 720)$ ,  $600p (800 \times 600)$  and  $480p (640 \times 480)$ . The horizontal axis represents the percentage of area occupied by the markers in each frame, and the vertical axis one indicates how many times ArUco3 is faster.

As can be observed, the proposed pipeline includes a
number of differences with respect to the original ArUco
pipeline that allows increasing significantly the processing
speed as we show next.

#### 405 4. Experiments and results

This section shows the results obtained to validate the methodology proposed for the detection of fiducial markers.

First, in Sect 4.1, the computing times of our proposal 409 are compared to the best alternatives found in the liter-410 ature: AprilTags [18], ChiliTags [36], ArToolKit+ [31], 411 as well as ArUco [17] which is included in the OpenCV 412 library<sup>3</sup>. Then, Sect. 4.2 analyses and compares the sensi-413 tivity of the proposed method with the above-mentioned 414 methods. The main goal is to demonstrate that our ap-415 proach is able to reliably detect the markers with a very 416 high true positive ratio, under a wide range of marker reso-417 lutions, while keeping the false positive rate to zero. After-418 ward, Sect. 4.3 studies the impact of the different system 419 parameters on the speed and sensitivity, while Sect. 4.4 420 evaluates the precision in the estimation of the corners. 421 Finally, Sect. 4.5 shows the performance of the proposed 422 method in a realistic video sequence with occlusions, illu-423 mination, and scale changes. 424

To carry out the first three experiments, several videos 425 have been recorded in our laboratory. Figure 4(b-e) shows 426 some images of the video sequences employed. For these 427 tests, a panel with a total of 16 markers was printed (Fig-428 ure 4a), four from each one of the fiducial markers em-429 ployed. The sequences were recorded at different distances 430 at a frame rate of 30 fps using an Honor 5 mobile phone at 431 4K resolution. The videos employed are publicly available 432 <sup>4</sup> for evaluation purposes. 433

In the video, there are frames in which the markers ap-434 pear as small as can be observed in Figure 4b, where 435 the area of each marker occupies only 0.5% of the image, 436 and frames in which the marker is observed as big as in 437 Figure 4e, where the marker occupies 40% of total im-438 age area. In total, the video sequences recorded sum up 439 to 10.666 frames. The video frames have been processed 440 at different resolutions so that the impact of the image 441 resolution in the computing time can be analyzed. In par-442 ticular, the following the standard image resolutions have 443 been employed: 4K ( $3840 \times 2160$ ), 1080p ( $1920 \times 1080$ ), 444 720p  $(1280 \times 720)$ , 600p  $(800 \times 600)$  and 480p  $(640 \times 480)$ . 445

All tests were performed using an Intel® Core <sup>TM</sup> i7-446 4700HQ 8-core processor with 8 GB RAM and Ubuntu 16.04 as the operating system. However, only one execution thread was employed in the tests performed. 449

It must be indicated that the code generated as part of this work has been publicly released as the version 3 of the popular ArUco library<sup>5</sup>. So, in the experiments section, the method proposed in this paper will be referred to as ArUco3.

#### 4.1. Speedup

This section compares the computing times of the proposed method with the most commonly used alternatives AprilTags, ArToolKit+, ChiliTags, and ArUco. To do so, we compute the speedup of our approach as the ratio between the computing time of an alternative  $(t_1)$  and the computing time of ArUco3  $(t_2)$  in processing the same image:

$$SpeedUp = t_1/t_2 \tag{7}$$

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In our method, the value  $\tau_c = 32$  was employed in all the sequences, while  $\tau_i$  and the segmentation threshold where automatically computed as explained in the Steps 2 and 7 of the proposed method (Sect. 3.2).

<sup>&</sup>lt;sup>3</sup>https://opencv.org/

<sup>&</sup>lt;sup>4</sup>https://mega.nz/#F!DnA1wIAQ!6f6owb81G0E7Sw3EfddUXQ

<sup>&</sup>lt;sup>5</sup>http://www.uco.es/grupos/ava/node/25

Table 1: Mean computing times (milliseconds) of the different steps of the proposed method for different resolutions.

	Resolution				
	480p	600p	720p	1080p	2160p
Step 1:Image Resize	0.037	0.050	0.057	0.068	0.101
Step 2:Image Segmentation	0.044	0.048	0.059	0.084	0.351
Step 3:Contour Extraction and Filtering	0.219	0.250	0.301	0.403	1.109
Step 4:Image Pyramid Creation	0.037	0.076	0.096	0.186	0.476
Step 5:Marker code extraction	0.510	0.519	0.542	0.547	0.583
Step 6:Corner Upsampling	0.058	0.065	0.079	0.096	0.134
Time (ms)	0.903	1.009	1.133	1.384	2.755

Fig. 5 shows the speedup of our approach for different 460 image resolutions. The horizontal axis represents the rel-461 ative area occupied by the marker in the image, while the 462 vertical axis represents the speedup. A total of 30 speed 463 measurements were performed for each image, taking the 464 median computing time for our evaluation. In the tests, 465 the speedup is evaluated as a function of the observed 466 marker area in order to better understand the behavior 467 of our approach. 468

The tests conducted clearly show that the proposed method (ArUco3) is faster than the rest of the methods and that the speedup increases with the image resolution and with the observed marker area. Compared to ArUco implementation in the OpenCV library, the proposed method is significantly faster, achieving a minimum speedup of 17 in 4K resolutions, up to 40 in the best case.

In order to properly analyze the computing times of the different steps of the proposed method (Sect. 3.2), Table 1 shows a summary for different image resolutions. Likewise, Fig. 6 shows the percentage of the total time required by each step. Please notice that Step 7 (Eq. 6) has been omitted because its computing time is negligible.

As can be seen, the two most time-consuming opera-482 tions are Step 3 and 5. In particular, Step 5 requires spe-483 cial attention, since it proves the validity of the multi-scale 484 method proposed for marker warping. It can be observed 485 in the table, that the amount of time employed by Step 5 486 is constant across all resolutions. In other words, the com-487 puting time does not increase significantly with the image 488 resolution. Also notice how the time of Step 3 increases 489 in 2160p. It is because this step involves operations that 490 depend on the image dimensions, which grow quadrati-491 cally. An interesting future work is to develop methods 492 reducing the time for contour extraction and filtering in 493 high-resolution images. 494

In any case, considering the average total computing time, the proposed method achieves in average more than 360 fps in 4K resolutions and more than 1000 fps in the lowest resolution, without any parallelism.





## 4.2. Sensitivity analysis

Correct detection of markers is a critical aspect that must be analyzed to verify that the proposed algorithm is able to obviate redundant information present in the scene, extracting exclusively marker information. Fig. 7 shows the True Positive Rate (TPR) of the proposed method as a function of the area occupied by the marker in the image for different image resolutions.

As can be observed, below certain marker area, the de-507 tection is not reliable. This is because the observed marker 508 area is very small, making it difficult to distinguish the 509 different bits of the inner binary code. Once the observed 510 area of the marker reaches a certain limit, the proposed 511 method achieves perfect detection in all resolutions. It 512 must be remarked, that the False Positive Rate is zero in 513 all cases tested. Since it is a binary problem, the True 514 Negative Rate is one (TNR=1-FPR). 515

For a comparative evaluation performance between 516 ArUco3 and the other methods, the TPR has been an-517 alyzed individually and the results are shown in Fig. 7. 518 As can be observed, ArUco behaves exactly like ArUco3. 519 AprilTags, however, has very poor behavior in all resolu-520 tions, especially as the marker or the image sizes increases. 521 As we already commented in Sect. 2, AprilTags does not 522 rely on warping the marker image but instead does a sub-523 sampling of a few pixels on the image in order to obtain 524 the binary code. This may be one of the reasons for its 525 poor performance. ArToolKit+ behaves reasonably well 526 across all the image resolutions and marker areas, while 527 Chilitags shows a somewhat unreliable behavior in all res-528 olutions but 480p. 529

In conclusion, the proposed approach behaves similar to the previous version of ArUco. 531



Figure 7: **True Positive Ratio.** Mean true positive ratio (TPR) for ArUco3, Chilitags, ArUco, ArToolKit+ and AprilTags for resolutions: 4K, 1080p, 720p, 600p and 480p), as function of the observed area for the set of markers.

#### 532 4.3. Analysis of parameters

The computing time and robustness of the proposed method depend mainly on two parameters, namely  $\tau_i$ which indicates the minimum size of the markers detected, and  $\tau_c$ , the size of the canonical image.

The parameter  $\tau_i$  has an influence on the computing 537 time, since it determines the size of the resized image  $I^r$ 538 (Eq. 1). We have analyzed the speed as a function of 539 this parameter and the results are shown in Fig. 8. The 540 figure represents the horizontal axis the value  $\tau_i$ , and in the 541 vertical axis, the average speed (measured as frames per 542 second) in the sequences analyzed, independently of the 543 observed marker area. A different line has been depicted 544 for each image resolution. In this case, we have set fixed 545 the parameter  $\tau_c = 32$ . 546

It can be observed that the curves follow a similar pat-547 tern in the five cases analyzed. In general, the maxi-548 mum increase in speed is obtained in the range of values 549  $\tau_i = (0, 0.2)$ . Beyond that point, the improvement be-550 comes marginal. To better understand the impact of this 551 parameter, Table 2 shows the reduction of the input im-552 age size I for different values of  $\tau_i$ . For instance, when 553  $\tau_i = 0.02$ , the resized image  $I^r$  is 48% smaller than the 554 original input image I (see Eq. 1). Beyond  $\tau_i = 0.2$ , the 555 resized image is so small that it has not a big impact in the 556 speedup because there are other steps with a fixed com-557

puting time such as the Step 5 (Marker Code Extraction). 55

Table 2: Image size reduction for different values of $\tau_i$ .						
$ au_i$	0.01	0.015	0.02	0.1	0.2	
Size reduction	0%	31%	48%	82%	90%	

In any case, it must be noticed that the proposed 559 method is able to achieve 1000 fps in 4K resolutions when 560 detecting markers larger than 10% ( $\tau_i = 0.1$ ) of the image area, and the same limit of 1000 fps is achieved for 1080p resolutions for  $\tau_i = 0.05$ . 563

With regards to the parameter  $\tau_c$ , it indirectly influences 564 the speed since it determines the size of the resized images 565 (Eq 1). The smaller it is, the smaller the resized image  $I^r$ . 566 Nevertheless, this parameter also has an influence on the 567 correct detection of the markers. The parameter indicates 568 the size of the canonical images used to identify the bi-569 nary code of markers. If the canonical image is very small, 570 pixels are mixed up, and identification is not robust. Con-571 sequently, the goal is to determine the minimum value of 572  $\tau_c$  that achieves the best TPR. Fig. 9 shows the TPR ob-573 tained for different configurations of the parameter  $\tau_c$ . As 574 can be seen, for low values of the parameter  $\tau_c$  (between 575 8 and 32) the system shows problems in the detection of 576 markers. However, for  $\tau_c \geq 32$  there is no improvement in 577 the TPR. Thus, we conclude that the value  $\tau_c = 32$  is the 578



Figure 8: **Parameter**  $\tau_i$ . Speed of method as a function of the parameter  $\tau_i$  for the different resolutions tested.



Figure 9: **Parameter**  $\tau_c$ . True positive rate obtained by different configurations of parameter  $\tau_c$ 



Figure 10: Vertex jitter measured for the different marker systems.

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#### 579 best choice.

#### 580 4.4. Precision of corner detection

An important aspect to consider in the detection of the 581 markers is vertex jitter, which refers to the noise in the 582 estimation of the corners' location. These errors are prob-583 lematic because they propagate to the estimation of the 584 camera pose. In our method, a corner upsampling step 585 (Step 6 in Sect. 3.2) is proposed to refine the corners' esti-586 mations from the reduced image  $I^r$  to the original image 587 I. This section analyzes the proposed method comparing 588 the results with the other marker systems. 589

In order to perform the experiments, the camera has 590 been placed at a fixed position recording the set of mark-591 ers already presented in Fig. 4a. Since the camera is not 592 moving, the average location estimated for each corner can 593 be considered to be the correct one (i.e., a Gaussian error 594 distribution is assumed). Then, the standard deviation is 595 an error measure for the localization of the corners. The 596 process has been repeated a total of six times at varying 597 distances and the results obtained are shown in Fig. 10 as 598 box plots. In Table 3, the average error of each method 599 has been indicated.

Table 3: Vertex jitter analysis: Standard deviations of the different methods in estimating the marker corners.

Method	ArUco	ArUco3	Chilitags	AprilTags	ArToolKit+
Average error (pix)	0.140	0.161	0.174	0.225	0.432

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As can be observed, the ArUco system obtains the best results, followed by our proposal ArUco3. However, it can be seen that the difference between both methods is of only 0.02 pixels, which is very small to consider it relevant. Chilitags shows a similar behavior than ArUco and ArUco3, but AprilTags and ArToolKit+ exhibit worse performance.

#### 4.5. Video sequence analysis

This section aims at showing the behavior of the pro-609 posed system in a realistic scenario. For that purpose, four 610 markers have been placed in an environment with irregular 611 lighting and a video sequence has been recorded using a 612 4K mobile phone camera. Figure 11(a-e) show the frames 613 1,665,1300,1700 and 2100 of the video sequence. At the 614 start of the sequence, the camera is around five meters 615 away from the markers. The camera approaches the mark-616 ers and then moves away again. As can be seen, around 617 frame 650 (Figure 11b), the user occludes the markers tem-618 porarily. 619

Figure 11f shows the values of the parameter  $\tau_i$  automatically calculated along the sequence and Figure 11g the processing speed. As can be observed, the system is able to automatically adapt the value of  $\tau_i$  according to the observed marker area, thus adapting the computing speed of the system. The maximum speed is obtained around the frame 1300 when the camera is closest to the markers.

It can also be observed that around frame 650 when the user occludes the markers with his hand, the system is unable to detect any marker. Thus, the system searches for the full resolution image ( $\tau_i = 0$ ) and the speed decreases. However, when the markers are observed again, the system recovers its speed.

Finally, Figure 11h shows the threshold values employed for segmentation in each frame. As can be seen, the system adapts to the illumination changes. Along the sequence, the system does not produce any false negative nor positives. 637

#### 5. Conclusions and future work

This paper has proposed a novel approach for detecting fiducial markers aimed at maximizing speed while preserving accuracy and robustness. The proposed method 641





Figure 11: Video Sequence in a realistic scenario. (a-e) Frames of the video sequence. The camera approaches the marker and then moves away. The user occludes the camera temporarily. (f) Evolution of the parameter  $\tau_i$  automatically computed. (g) Speed of the proposed method in each frame of the sequence. (h) Thresholds automatically computed for each frame. The system adapts to illumination changes.

is specially designed to take advantage of the increasing 642 camera resolutions available nowadays. Instead of detect-643 ing markers in the original image, a smaller version of the 644 image is employed, in which the detection can be done 645 at higher speed. By wisely employing a multi-scale image 646 representation, the proposed method is able to find the po-647 sition of the marker corners with subpixel accuracy in the 648 original image. The size of the processed image, as well 649 as the threshold employed for segmentation, are dynam-650 ically adapted in each frame considering the information 651 of the previous one. As a consequence, the system speed 652 dynamically adapts in order to achieve the maximum per-653 formance. 654

As shown experimentally, the proposed method outperforms the state-of-the-art systems in terms of computing speed, without compromising the sensitivity or the precision. Our method is between 17 and 40 times faster than the ArUco approach implemented in the OpenCV library. When compared to other approaches such as Chilitags, AprilTags, and ArToolKit+, our method achieves even higher speedups.

We consider as possible future works to investigate the use of the proposed method in fish-eye cameras. The idea is to compare the method with the rectified images if there is analyze the method's performance in presence of high distortion. Also, we as well as to characterize the performance when multiple fiducial markers with significantly different scales are present in the same image.

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Our system, which is publicly available as open source code<sup>6</sup>, is a cost-effective tool for fast and precise self-localization in applications such as robotics, unmanned vehicles or augmented reality applications. 673

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<sup>&</sup>lt;sup>6</sup>http://www.uco.es/grupos/ava/node/25

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