Lecture: Eye Camera Settings and Calibration

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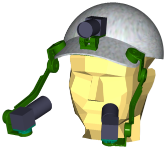

DIY Eye Tracking

- To fully understand how it works, build it yourself
- Babcock and Pelz (2004) were one of the first to describe a DIY wearable analog video tracker
  - an expensive video multiplexer was needed to synchronize eye and scene cameras
  - system relied on somewhat dated (by today's standards) video recorder (a Sony DCR-TRV19 DVR)
  - nevertheless, fostered nascent open-source movement

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General Motion Capture

An eye tracker is a specialized kind of motion capture (*mocap*) device, delineated within four classes:

- **Electromechanical:** skin sensors around the eye cavity (electro-oculography, or EOG)
- **Electromagnetic:** a metallic stalk is fixed to contact lens
- **Video-oculography, or VOG:** analysis (historically manual) of video frames
- **Video-based with IR reflection:** eye trackers benefit from one (or more) of four types of IR reflections, two from the front and rear surface of the cornea, two from the front and rear surface of the crystalline lens; all four reflections are known as the Purkinje images (Crane, 1994)

For comparison with video mocap, see Menache (2000)

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Two Approaches to Video-Based Eye Tracking

1. Based on a 3D eyeball model (e.g., Gullstrand's), compute the ray emanating from a central (nodal) point within the eye, then compute ray/plane intersections within the scene,

$$p = c + t \cdot L,$$

   where p is the 3D Point Of Gaze, or POG, c is the center of the eye (or cornea, modeled by a sphere), t is the parametric distance along the ray L coinciding with the optical or visual axis
2. Based on traditional video-oculography, estimate the limbus center (x, y)—the POG in the scene image is obtained via a mapping of (x, y) to scene coordinates (s_x, s_y) given known calibration point coordinators

- See Hennessey (2008) for a detailed review

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Building Our Own

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- With digital cameras becoming affordable and easier to use, it became feasible (and fairly easy)
- Derrick Parkhurst suggested a pair of cameras to use
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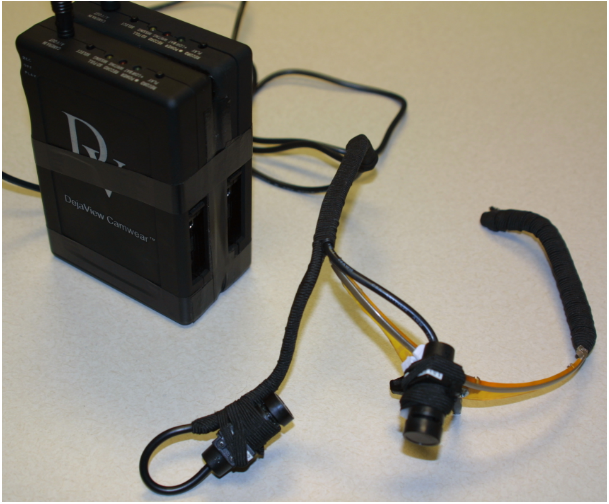
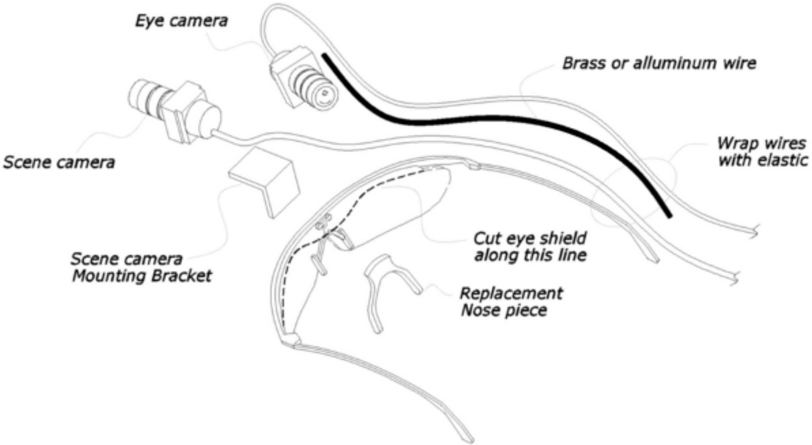
Off-line Visible Spectrum Wearable Tracker



Visible Spectrum Wearable Schematic



Parts List

Vendor	Description	Price
AOSafety	X-Factor XF503 safety glasses	$10.00
AOSafety	I-Pilot 90714 nose piece	$10.00
Jose Ann Fabric	Polyester braided elastic	$1.50
Home Depot	Screws	$0.75
Amazon.com	Aluminum pod 1/4″ dia. 18″ long	$5.00
DejaView	Two Camwear Model 200 cameras	$500.00
	Total	$527.25

Visible Spectrum Design

- Design based to a large extent on that of Li et al. (2006)
- We first explored tracking in the visible spectrum, following Li and Parkhurst (2006)
- Gaze tracking relies on tracking of the limbus, or the iris/sclera boundary
- No IR simplifies design but complicates image analysis
- Headgear must be stable to offer sufficiently stable reference point for gaze point estimation

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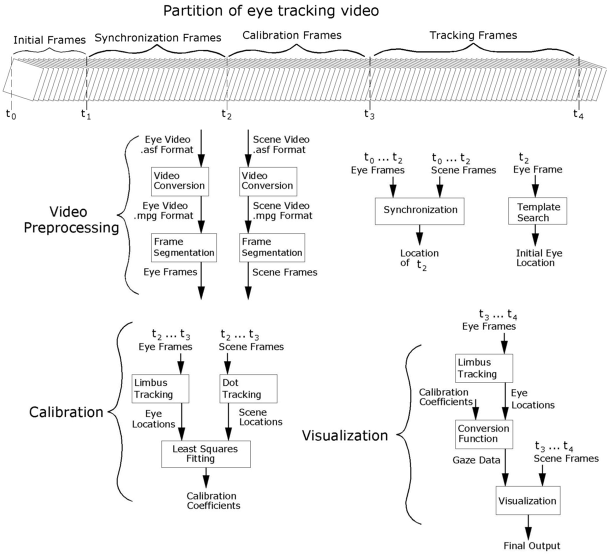
Software Architecture



Basic Steps

1. Video preprocessing: convert to some known format from which you can extract frames
   - we had problem with `.asf` format; turns out it was "proprietary" by virtue of 4 unknown bytes in file header, easily "fixed" once known
   - finding good open source codec library was a problem; latest version of `ffmpeg` is excellent
2. Synchronization: line up eye and scene video feeds via "flash screen" (as suggested by Li and Parkhurst (2006))
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Pupil Tracking

- Major problem is pupil tracking—a good starting point is the *Starburst* algorithm (Li et al., 2006)
- Idea is to shoot out rays from within pupil to locate limbic boundary feature points along those rays
- A seed point is needed from which to shoot rays
- Our initial approach was based on template matching:
  - Canny's (1986) edge detector
  - Borgefors' (1988) chamfer image
  - pixel pattern

(a) image frame (b) Canny edges (c) chamfer image (d) pattern

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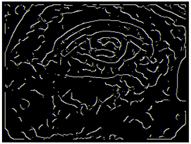

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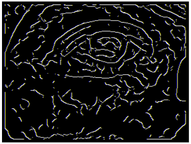
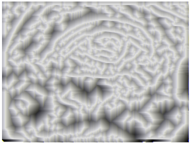


(b) Canny edges



(c) chamfer image

(d) pattern

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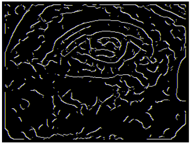

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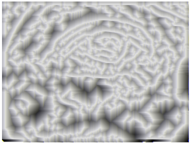
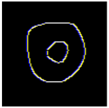


(c) chamfer image



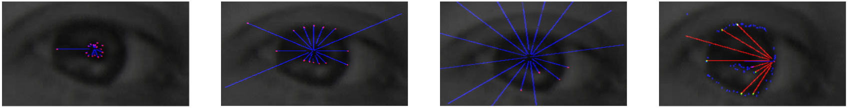
(d) pattern

Searchmap

- A gradient image is also use to guide the algorithm for finding Starburst's initial seed point

```
MakeSearchMap(Gradient image G, Chamfer image C,
              Template image T, Template gradient GT)
{
  for all template locations x,y {
    sum = 0;
    for all locations dx,dy ∈ T ≠ 0
      sum += G(x + dx, y + dy) · GT(dx, dy) × C(x + dx, y + dy);
    M(x, y) = sum; // M is the search map;
  }
  return maximum pixel in M;
}
```

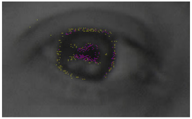
Feature Point Detection (Starburst)



FeatureDetect(Gradient image G, Threshold T, Start point S)
{
  for θ **from** 0 **to** 2π { // phase 1
    $L = S$; // L is the current location
    $\Delta_x = \cos(\theta)$; $\Delta_y = \sin(\theta)$;
    while $G(L) \cdot \Delta < T$ { $L(x)$ += Δ_x; $L(y)$ += Δ_y; }
    push L **on** P; // P is a point stack
  }
  for θ **from** 2π **to** 0 { // phase 2
    pop L **off** P; **add** L **to** P'; // P' is a point set
    for α **from** $-(\theta + \frac{\pi}{6})$ **to** $-(\theta - \frac{\pi}{6})$
      while $G_L \cdot \Delta < T$ { $L(x)$ += Δ_x; $L(y)$ += Δ_y; }
    add L **to** P';
  }
  return P';
}

Ellipse Fitting

- Starburst returns a set of feature points
- Next step is to fit ellipses to points
- Starburst used RANSAC (Random Sample Consensus)
- Our visible spectrum approach fit multiple ellipses and distinguished between those that fit the pupil, those that fit the limbus, and those that spanned both (erroneous)
- Ellipse is used to estimate pupil center (x, y)



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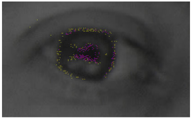

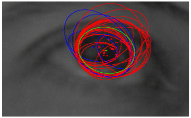

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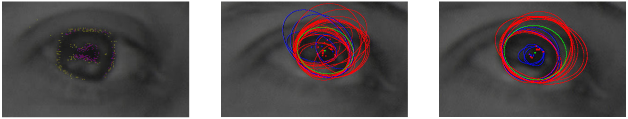

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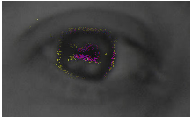

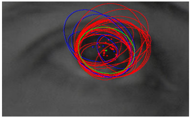

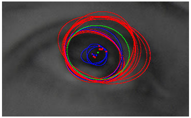

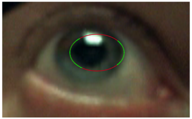

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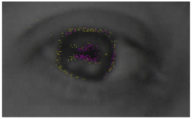
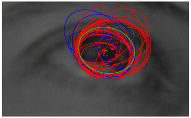
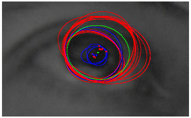
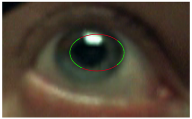
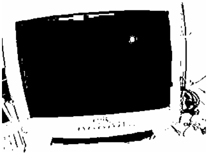





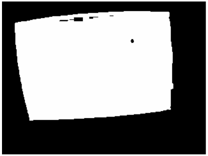
Calibration

- Calibration maps pupil center (x, y) in eye frames to calibration points (s_x, s_y) in the scene frames
- Calibration dots are displayed on a computer monitor viewed by the scene camera, tracked by a 5-step algorithm

(a) scene frame



(b) low threshold



(c) get largest region

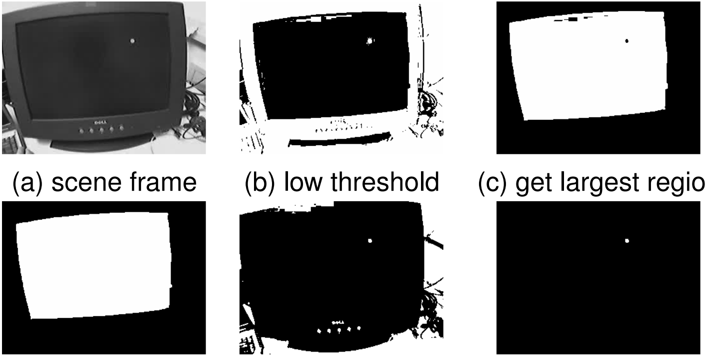
(d) flood fill holes

(e) high threshold

(f) final result

Calibration

- Calibration maps pupil center (x, y) in eye frames to calibration points (s_x, s_y) in the scene frames
- Calibration dots are displayed on a computer monitor viewed by the scene camera, tracked by a 5-step algorithm



(a) scene frame (b) low threshold (c) get largest region

(d) flood fill holes (e) high threshold (f) final result

Computing the Calibration Homography

- With calibration screen coordinates (s_x, s_y) and pupil center (x, y), the mapping is expressed as a second order polynomial (C. H. Morimoto & Mimica, 2005):

$$\begin{aligned} s_x &= a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 \\ s_y &= b_0 + b_1 x + b_2 y + b_3 xy + b_4 x^2 + b_5 y^2 \end{aligned}$$

- The parameters $\{a_0, \ldots, a_5\}$ and $\{b_0, \ldots, b_5\}$ can be computed via Lagrange's method of least squares (e.g, see Lancaster and Šalkauskas (1986))

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Computing the Calibration Homography

- Reformulate the above equations by writing:

$$\begin{bmatrix} s_{1x} & s_{1y} \\ s_{2x} & s_{2y} \\ \vdots & \vdots \\ s_{ix} & s_{iy} \\ \vdots & \vdots \\ s_{nx} & s_{ny} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & xy_1 & x_1^2 & y_1^2 \\ 1 & x_2 & y_2 & xy_2 & x_2^2 & y_2^2 \\ \vdots & \vdots & \vdots & & & \\ 1 & x_i & y_i & xy_i & x_i^2 & y_i^2 \\ \vdots & \vdots & \vdots & & & \\ 1 & x_n & y_n & xy_n & x_n^2 & y_n^2 \end{bmatrix} \begin{bmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \\ a_5 & b_5 \end{bmatrix}$$

or in matrix notation,

$$\mathbf{Y} = \mathbf{X}\hat{\mathbf{B}}$$

- An estimate of $\hat{\mathbf{B}}$ is obtained by left-multiplying by $(\mathbf{X}^T\mathbf{X})^{-1}$

$$\hat{\mathbf{B}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$$

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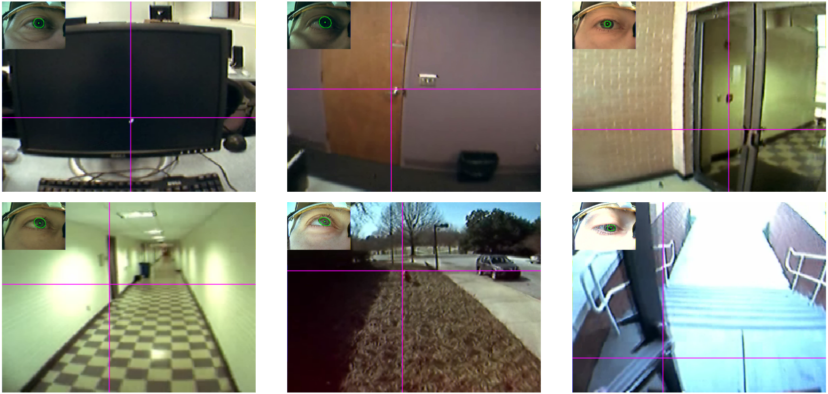
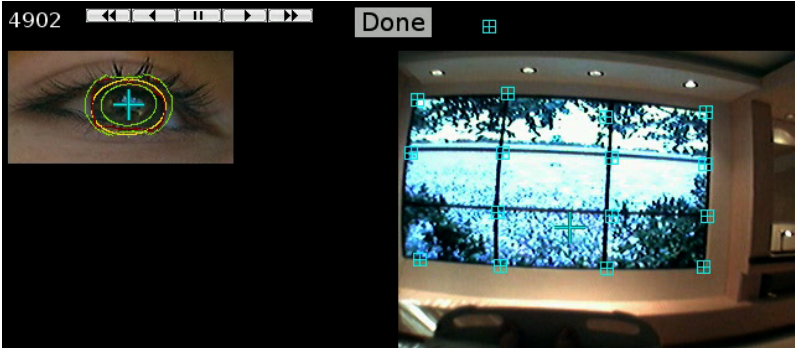
Preliminary Results



Natural Tasks



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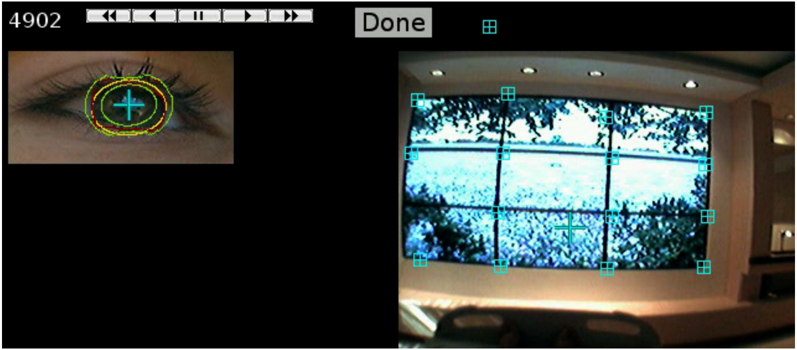

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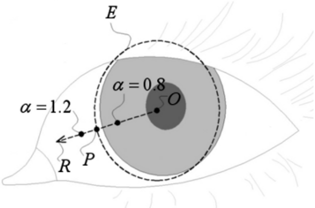
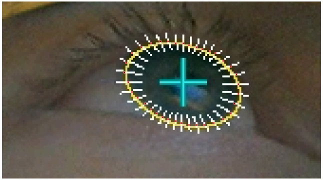
Tracking Refinements





- Cast ray R from origin O
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- max ∇ signifies feature point
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  - assume points are near E identified in previous frame
  - if P is intersection of R and E constrain search via

$$\max \nabla(O + \alpha(P - O)) \; : \; 0.8 < \alpha < 1.2)$$

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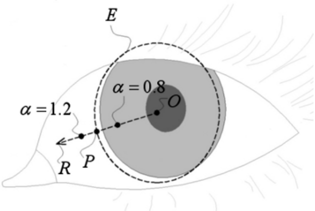

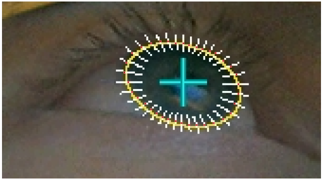

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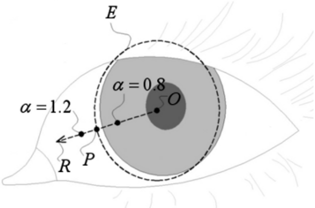

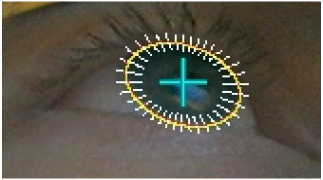

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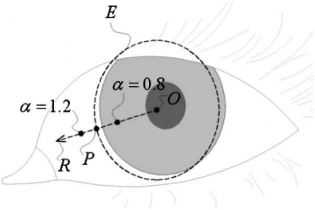

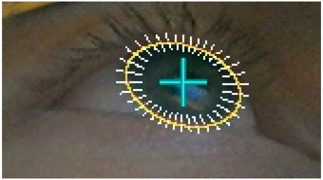

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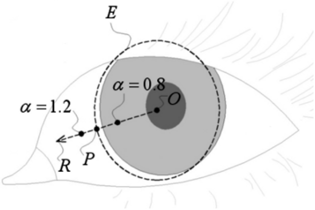

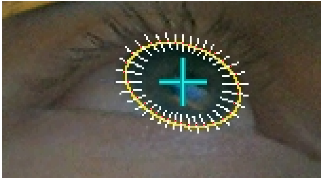

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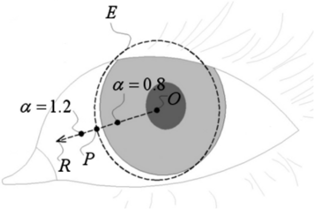

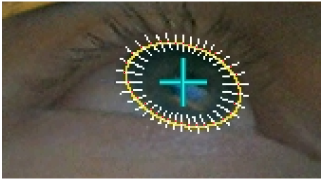

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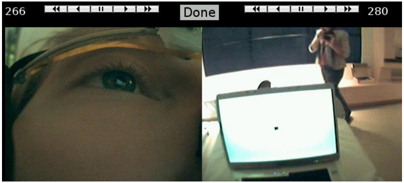
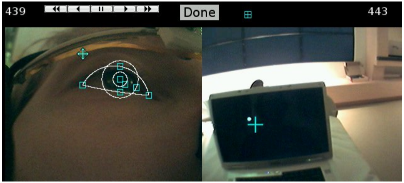
Interactive Manual Intervention





- Manual intervention via GUI to synchronize video streams (find flashes in image frames)
- Seed Starburst via manually overlayed template
- Similarly, use crosshair for initial location of calibration dot

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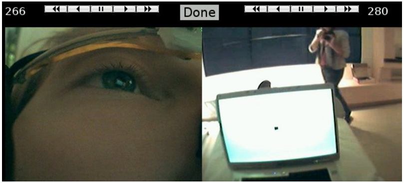

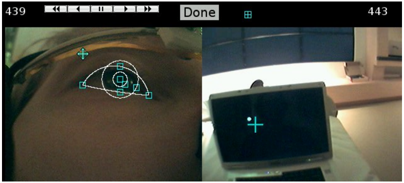

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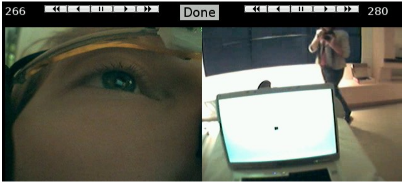

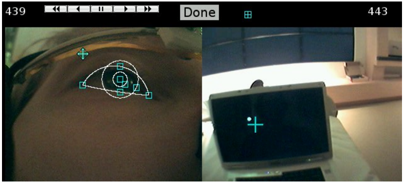

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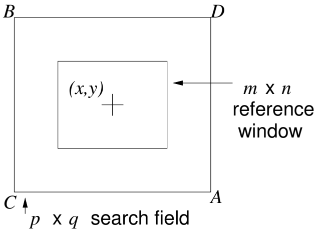
AOI Tracking



- Need to track AOIs to count fixations falling within
- Similar to *match-moving* trackbox (Paolini, 2006)
- Bright spot is found where $I(x, y) - \mu$ is maximum
- For efficiency, a summed area table S is precomputed (Crow, 1984; Viola & Jones, 2004)

$$S(x, y) = \sum_i \sum_j I(i, j), \quad 0 < i < x, \quad 0 < j < y$$

$$\mu = \frac{(S(A) + S(B)) - (S(C) + S(D))}{p \times q}$$

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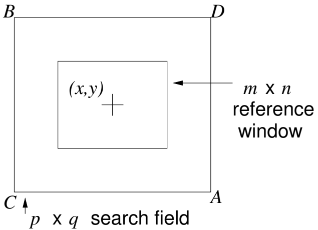

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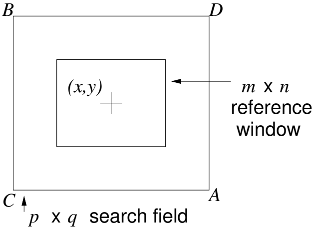

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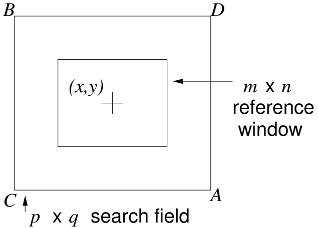

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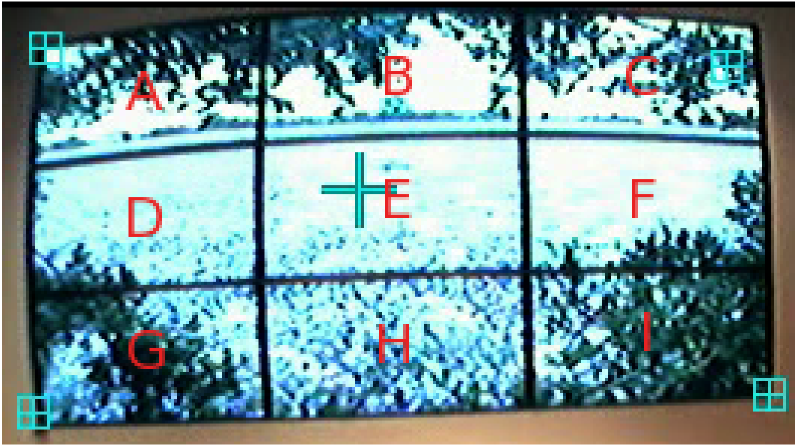
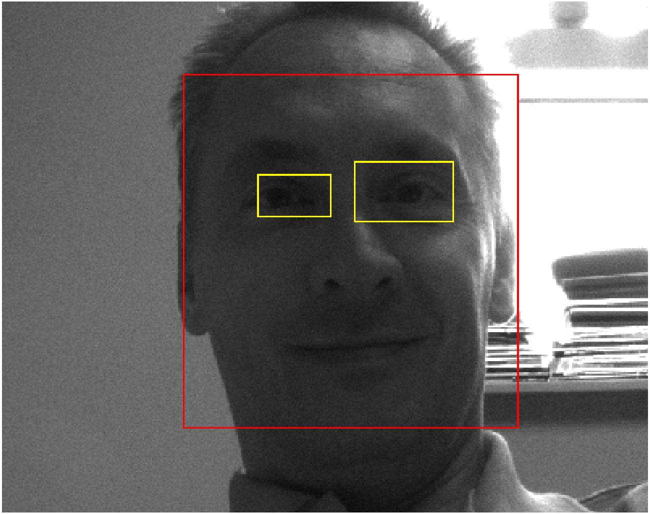
Applied Example



- Technique worked fairly well in applied setting

OpenCV



- We are exploring OpenCV for feature tracking

Revisiting Ellipse Fitting

- Ellipse fitting is an essential eye tracking component
- RANSAC fits many ellipses, one winner—slow
- Given n points $\{x_i, y_i\}$, fit the quadratic such that

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

- We can rewrite this as

$$\begin{bmatrix} x^2 & xy & y^2 & x & y & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d & e & f \end{bmatrix}^T = 0$$

- Then rewrite in matrix notation

$$\mathbf{XA} = \mathbf{B}$$

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Ellipse definition

- Alternate representation of ellipse is

$$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{s^2} = 1$$

- With center (h, k) and axes $(\mathbf{r}, \mathbf{s})$ of length $r = \|\mathbf{r}\|$ and $s = \|\mathbf{s}\|$ respectively, coefficients are expressed as:

$$\begin{aligned}
a &= s^2M^2 + r^2N^2 \\
b &= 2MN(s^2 - r^2) \\
c &= s^2N^2 + r^2M^2 \\
d &= -2h(s^2M^2 + r^2N^2) - 2MNk(s^2 - r^2) \\
e &= -2k(s^2N^2 + r^2M^2) - 2MNh(s^2 - r^2) \\
f &= M^2(s^2h^2 + r^2k^2) + N^2(r^2h^2 + s^2k^2) + 2MNhk(s^2 - r^2) - r^2s^2
\end{aligned}$$

- With $M = \cos(\theta)$, $N = \sin(\theta)$ for an ellipse rotated about its center by angle $\theta = \tan^{-1}(\mathbf{r}_y/\mathbf{r}_x)$

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f &= M^2(s^2h^2+r^2k^2)+N^2(r^2h^2+s^2k^2)+2MNhk(s^2-r^2)-r^2s^2
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- With $M=\cos(\theta)$, $N=\sin(\theta)$ for an ellipse rotated about its center by angle $\theta=\tan^{-1}(\mathbf{r}_y/\mathbf{r}_x)$

Ellipse definition

- Alternate representation of ellipse is

$$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{s^2} = 1$$

- With center (h, k) and axes $(\mathbf{r}, \mathbf{s})$ of length $r = \|\mathbf{r}\|$ and $s = \|\mathbf{s}\|$ respectively, coefficients are expressed as:

$$\begin{aligned}
a &= s^2M^2 + r^2N^2 \\
b &= 2MN(s^2 - r^2) \\
c &= s^2N^2 + r^2M^2 \\
d &= -2h(s^2M^2 + r^2N^2) - 2MNk(s^2 - r^2) \\
e &= -2k(s^2N^2 + r^2M^2) - 2MNh(s^2 - r^2) \\
f &= M^2(s^2h^2 + r^2k^2) + N^2(r^2h^2 + s^2k^2) + 2MNhk(s^2 - r^2) - r^2s^2
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Method of Least-Squares

- We can set $f = 1$ (or any non-zero arbitrary constant),

$$ax^2 + bxy + cy^2 + dx + ey = -1$$

- Then rewrite as

$$\begin{aligned} \begin{bmatrix} x^2 & xy & y^2 & x & y \end{bmatrix} \begin{bmatrix} a & b & c & d & e \end{bmatrix}^T &= -f \\ \mathbf{XA} &= \mathbf{B} \end{aligned}$$

- And solve using the pseudo-inverse of $\mathbf{X}$

$$\mathbf{A} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{B}$$

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Fitzgibbon et al. (1996)

- Most eye tracking papers refer to Fitzgibbon et al. (1996)
- Method exploits ellipse-specific constraint $b^2 - 4ac < 0$
- The fitting problem is reformulated as

$$\min_{\mathbf{a}} \|\mathbf{D}\mathbf{a}\|^2 \quad \text{subject to} \quad \mathbf{a}^T\mathbf{C}\mathbf{a} = 1 \tag{1}$$

where the *design matrix* $\mathbf{D}$ of size $n \times 6$,

$$\mathbf{D} = \begin{pmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ \vdots & & & & & \\ x_i^2 & x_iy_i & y_i^2 & x_i & y_i & 1 \\ \vdots & & & & & \\ x_n^2 & x_ny_n & y_n^2 & x_n & y_n & 1 \end{pmatrix}$$

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$$\mathbf{C} = \begin{pmatrix} 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- The minimization problem (1) is solved by a quadratically constrained least squares minimization
- Apply the Lagrange multipliers for the optimal solution **a**

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represents the least squares minimization in which the operator S denotes the sum

$$S_{x^ay^b} = \sum_{i=1}^{n} x_i^a y_i^b$$

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Halíř and Flusser (1998)

- Halíř and Flusser (1998) break up the design matrix **D** into its quadratic and linear components

$$\mathbf{D} = (\mathbf{D}_1 \mid \mathbf{D}_2)$$

where

$$\mathbf{D}_1 = \begin{pmatrix} x_1^2 & x_1 y_1 & y_1^2 \\ \vdots & & \\ x_i^2 & x_i y_i & y_i^2 \\ \vdots & & \\ x_n^2 & x_n y_n & y_n^2 \end{pmatrix} \quad \text{and} \quad \mathbf{D}_2 = \begin{pmatrix} x_1 & y_1 & 1 \\ \vdots & & \\ x_i & y_i & 1 \\ \vdots & & \\ x_n & y_n & 1 \end{pmatrix}$$

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- Next, the scatter matrix **S** is split as follows

$$\mathbf{S} = \left(\begin{array}{c|c} \mathbf{S}_1 & \mathbf{S}_2 \\ \hline \mathbf{S}_2^T & \mathbf{S}_3 \end{array}\right) \quad \text{with} \quad \begin{aligned} \mathbf{S}_1 &= \mathbf{D}_1^T \mathbf{D}_1 \\ \mathbf{S}_2 &= \mathbf{D}_1^T \mathbf{D}_2 \\ \mathbf{S}_3 &= \mathbf{D}_2^T \mathbf{D}_2 \end{aligned}$$

- The constraint matrix **C** is expressed similarly

$$\mathbf{C} = \left(\begin{array}{c|c} \mathbf{C}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \quad \text{where} \quad \mathbf{C}_1 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

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$$\mathbf{a} = \begin{pmatrix} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \end{pmatrix} \quad \text{where} \quad \mathbf{a}_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

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- From (4), solve for $\mathbf{a}_2$

$$\mathbf{a}_2 = -\mathbf{S}_3^{-1}\mathbf{S}_2^T\mathbf{a}_1 \quad (5)$$

- and plug (5) into (3) to yield

$$\left(\mathbf{S}_1 - \mathbf{S}_2\mathbf{S}_3^{-1}\mathbf{S}_2^T\right)\mathbf{a}_1 = \lambda\mathbf{C}_1\mathbf{a}_1 \quad (6)$$

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- Because $\mathbf{C}_1$ is regular, (6) can be rewritten as

$$\mathbf{C}_1^{-1}\left(\mathbf{S}_1 - \mathbf{S}_2\mathbf{S}_3^{-1}\mathbf{S}_2^T\right)\mathbf{a}_1 = \lambda\mathbf{a}_1 \qquad (7)$$

- where $\mathbf{M} = \mathbf{C}_1^{-1}\left(\mathbf{S}_1 - \mathbf{S}_2\mathbf{S}_3^{-1}\mathbf{S}_2^T\right)$ is the *reduced scatter matrix* of size 3×3
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$$\mathbf{C}_1^{-1}\left(\mathbf{S}_1 - \mathbf{S}_2\mathbf{S}_3^{-1}\mathbf{S}_2^T\right)\mathbf{a}_1 = \lambda\mathbf{a}_1 \tag{7}$$

- where $\mathbf{M} = \mathbf{C}_1^{-1}\left(\mathbf{S}_1 - \mathbf{S}_2\mathbf{S}_3^{-1}\mathbf{S}_2^T\right)$ is the *reduced scatter matrix* of size 3×3
- For the optimal fit, it is enough to find the appropriate eigenvector $\mathbf{a}_1$ of matrix $\mathbf{M}$
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Further enhancements

- For improved numerical stability, Paul and Zsombor-Murray (2004) suggest first translating all data points then scaling isotropically to ensure $\sqrt{2}$ root mean square distance of the points to the origin, i.e., using the transformation matrix $\mathbf{T}$

$$\mathbf{T} = \begin{pmatrix} s & 0 & -s\overline{x} \\ 0 & s & -s\overline{y} \\ 0 & 0 & 0 \end{pmatrix}$$

where

$$s = \frac{n\sqrt{2}}{\sum_{i=1}^{n} \sqrt{(x_i - \overline{x})^2 + (y_i - \overline{y})^2}}$$

Further Enhancements

- After fitting, Paul and Zsombor-Murray (2004) suggest transforming data back to the original position and scaling via

$$\mathbf{K}^* = \mathbf{T}^T\mathbf{K}\mathbf{T}$$

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$$m = \sqrt{\frac{2n}{\sum_{i=1}^{n} (x_i - \overline{x})^2 + (y_i - \overline{y})^2}}$$

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Implementation Notes

- Eventual solution obtained “officially” from the IEEE PAMI paper by Fitzgibbon et al. (1999), but practically speaking from their web page:

`http://research.microsoft.com/en-us/um/people/awf/ellipse/`

  containing, you guessed it, MATLAB code
- They also normalize the points by shifting to the centroid and scaling, i.e.,

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$$\mathbf{s} = \begin{bmatrix} \frac{\max(x_i) - \min(x_i)}{2} & \frac{\max(y_i) - \min(y_i)}{2} \end{bmatrix}$$

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- What's really useful about Fitzgibbon et al.'s MATLAB code is that it includes the steps to "unnormalize" the fitted ellipse and to recover the ellipse parameters (h, k), (r, s), and θ
- Only problem is this line:

```
[evec_x, eval_x] = eig(inv(tmpD) * (tmpA - tmpB*tmpE));
```

which is the general solution to the eigensystem represented by

$$\mathbf{M} = \mathbf{C}_1^{-1}\left(\mathbf{S}_1 - \mathbf{S}_2\mathbf{S}_3^{-1}\mathbf{S}_2^T\right)$$

which happens to be non-symmetric
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Obtaining eigenvectors from non-symmetric matrix

- My go-to book for this sort of thing states:

  > *We consider the problem of finding* all *eigenvectors of a nonsymmetric matrix as lying beyond the scope of this book.*

  (Numerical Recipes in C, 2nd ed., Press et al., 1992)
- What's involved?
  1. balancing the matrix
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TNT and JAMA

- Luckily, we have Google!
- It came up with the Template Numerical Toolkit, made freely available by NIST here:

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- This has both TNT, templated C++ arrays, as well as JAMA
- JAMA/C++ is a translation of the Java Matrix Library, developed by the Mathworks and NIST, into C++ classical matrix linear algebra, and includes:
  - LU
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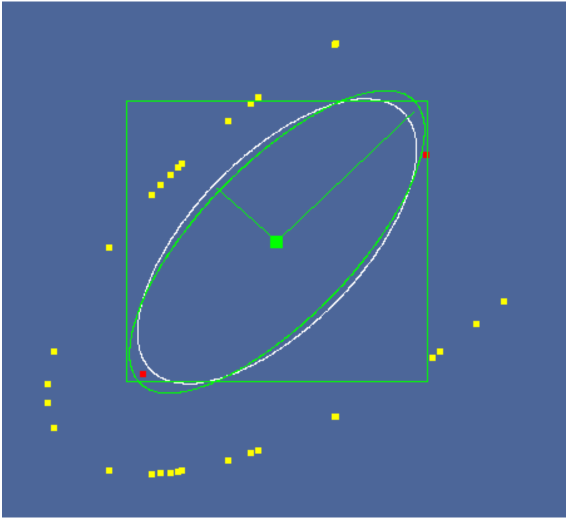
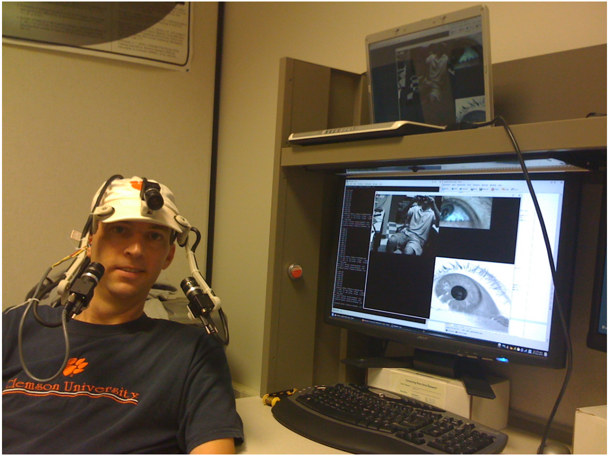
- The last piece of the puzzle I needed was the C++ equivalent to the `eig()` MATLAB call:

```
JAMA::Eigenvalue<double> eigen(TNT_M);
eigen.getV(TNT_V);
eigen.getRealEigenvalues(TNT_E);
```

Demo



On-line NIR Spectrum Wearable Tracker



Parts List

Vendor	Part No.	Desc.	Quant.	Unit Cost
Point Grey	FL2-03S2C	Scene Camera (c)	1	$695
Point Grey	FL2-03S2M	Eye Cameras (b/w)	2	$695
Point Grey	FWB-EC-2PORT	1394 ExpressCard	1	$80
Point Grey	HF12.5HA-1B	Eye Camera Lenses	2	$170
Point Grey	DF6HA-1B	Scene Camera Lens	1	$190
Point Grey	ACC-01-2012	1394b Cables	3	$150
Point Grey	FW-HUB-5PORT	1394b Hub	1	$139
UKA Optics	IR83M25.5	IR filter	2	$185
QuickParts	Custom Order	Plastic Mounts		$476
Various		LED, misc. parts		$50
			Total	$4180

Courtesy of Lafayette Instruments, Inc.

Current Features

- On-line, real-time processing
- IEEE 1394b (`libraw1394, libdc1394`)
  - DMA access to cameras
  - Format-7 allows specification of hardware ROI
- Video encoding/decoding (`ffmpeg`)
- Direct ellipse fitting (Fitzgibbon et al., 1999)

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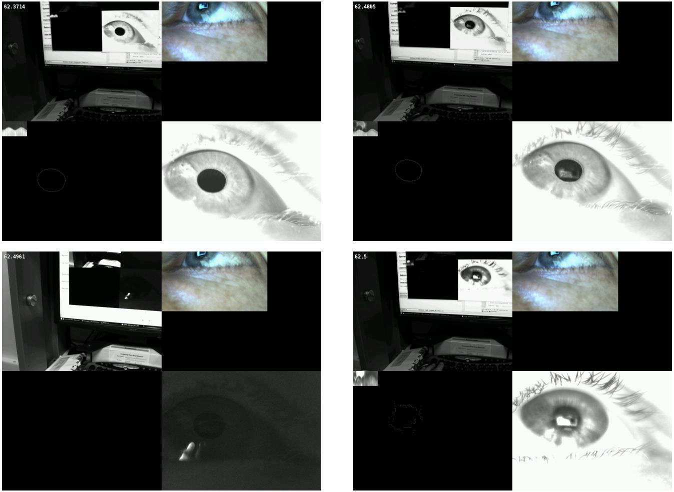
Why NIR?

IR filter | no IR filter

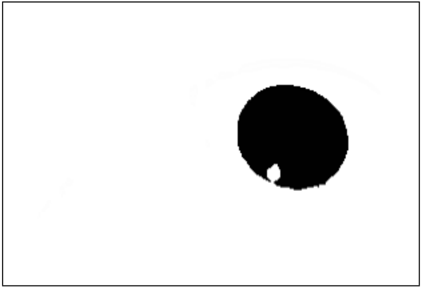
NIR LED

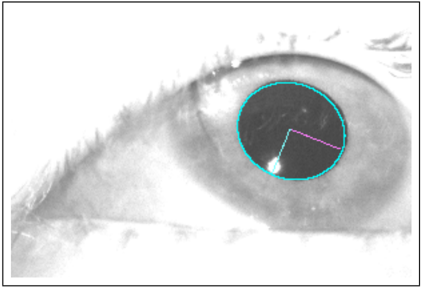
no NIR LED



NIR

- NIR illumination: 940 nm LED, radiant power output 16 mW
- Simplifies pupil detection (thresholding)
- Allows frame-differencing method as proposed by Ebisawa and Satoh (1993), popularized by C. Morimoto et al. (2000)





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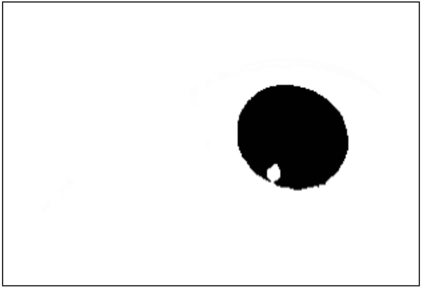

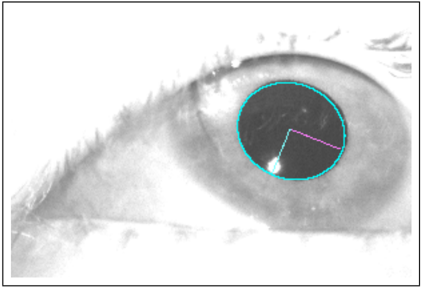

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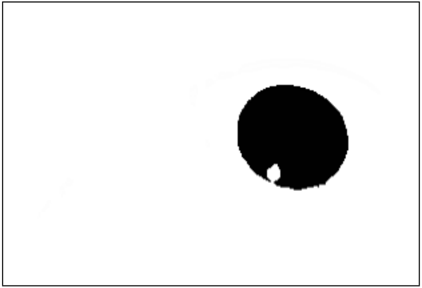
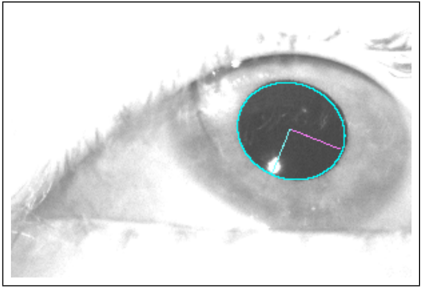
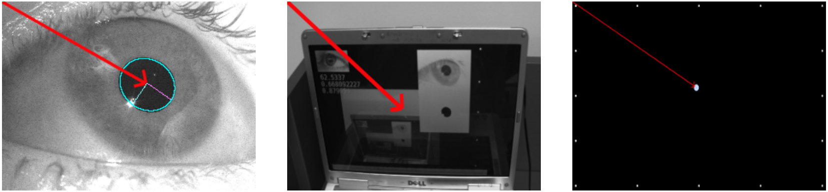




Calibration



- Calibration relies on image frame origin
  (we should really be using corneal reflection)
- Using laptop screen to display calibration dots

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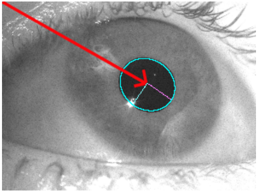

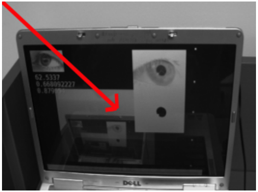

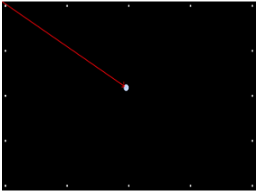

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- Keep in mind importance of IR light
- Watch for spurious reflections (mascara, spectacles, etc.)
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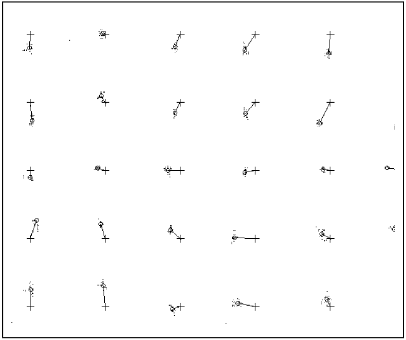

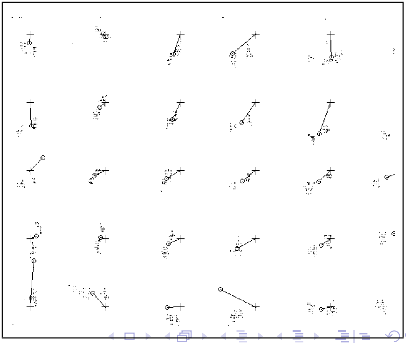

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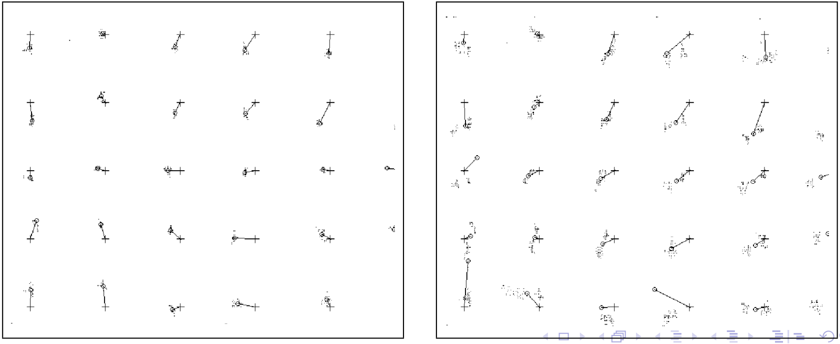

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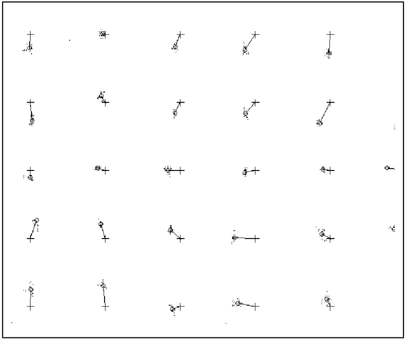

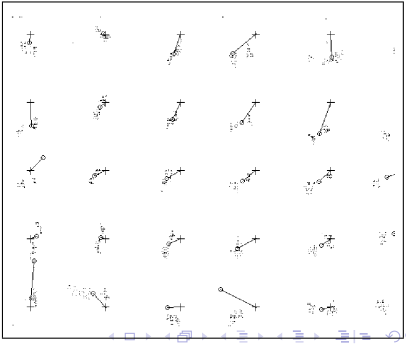

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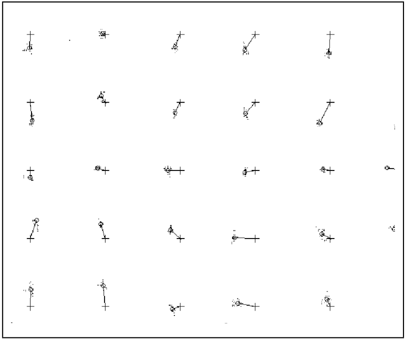

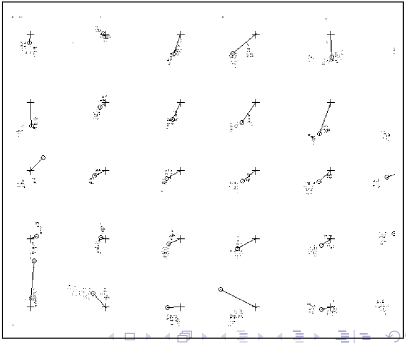

Summary

- Keep in mind importance of IR light
- Watch for spurious reflections (mascara, spectacles, etc.)
- As the late Prof. Stark said, “Calibrate, calibrate, calibrate!”
    - technology has improved since Larry’s time, but . . .
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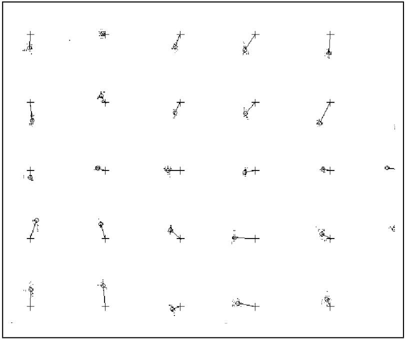

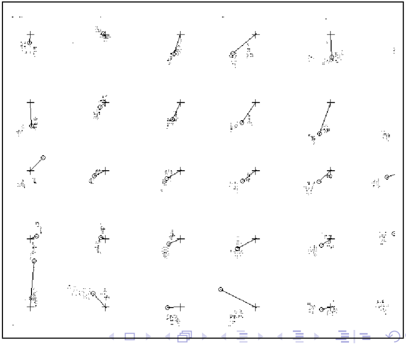

To DIY or not to DIY?

- If you want to build it yourself, use good cameras, e.g.,
  - IEEE 1394b Digital Cameras support 80+ fps with hardware ROI (format7)
  - e.g., Flea2 cameras from Point Grey Research
  - be careful about built-in IR filter—b/w Flea2s don't have one, color Flea2s do
  - choose correct lens FOV, e.g., we have
    - 6mm (wide angle for scene)
    - 12.5mm (medium range)
    - 25mm (macro-like for close range)
  - IR lens filters seem difficult to obtain, but not impossible
  - PGR cameras also come with powered "pigtail" that let you easily connect IR LEDs

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Comments, Questions, Discussion?

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